Fourier Analysis

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Review

Thm. Let $f \in M(\mathbb{R})$. Suppose that $\hat{f} \in M(\mathbb{R})$.

Then

$$f(x) = \int_{\mathbb{R}} \hat{f}(\xi)\, e^{2\pi i \xi x}\, d\xi.$$ (Fourier Inversion Formula)

and

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2\, d\xi$$

(Plancherel formula)

Application 1: Time-dependent heat equation on the real line

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}, & x \in \mathbb{R},\ t > 0 \\ u(x,0) = f(x). \end{cases}$$

Define

$$\mathcal{H}_t(x) = \frac{1}{\sqrt{4\pi t}}\, e^{-\frac{x^2}{4t}}, \quad x \in \mathbb{R},\ t > 0$$

(heat kernel on the real line)

$\left(\widehat{\mathcal{H}_t}(\xi) = e^{-4\pi^2 \xi^2 t} \right)$

Let $\mathcal{S}(\mathbb{R})$ denote the Schwartz space, i.e. the

Collection of $\mathbb{C}$-valued functions $f \in C^\infty(\mathbb{R})$ such that

$$\sup_{x\in\mathbb{R}} \left| x^k f^{(\ell)}(x) \right| < \infty \qquad \forall\ k, \ell \geq 0 .$$

Thm A Let $f \in S(\mathbb{R})$. Let

$$u(x,t) = f * \mathcal{H}_t(x).$$

Then

① $u \in C^\infty(\mathbb{R}\times\mathbb{R}_+)$, $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$ on $\mathbb{R}\times\mathbb{R}_+$ $= (-\infty,\infty)\times(0,\infty)$

② $u(x,t) \rightrightarrows f(x)$ as $t\to 0$

③ $\displaystyle\int_{\mathbb{R}} \left| u(x,t) - f(x) \right|^2 dx \to 0$ as $t \to 0$

Pf. Part ① was proved in the last lecture. Here we prove ② and ③.

Notice that $u(x,t) = f * \mathcal{H}_t(x)$.

Since $\{\mathcal{H}_t\}_{t>0}$ is a good kernel on $\mathbb{R}$ as $t \to 0$, so ② is a direct consequence of the convergence theorem for good kernels.

To prove ③, we use the Plancherel formula.

Notice that

$$\int_{-\infty}^{\infty} |u(x,t) - f(x)|^2 dx$$

$$= \int_{-\infty}^{\infty} |\hat{u}(\xi,t) - \hat{f}(\xi)|^2 d\xi$$

$$= \int_{-\infty}^{\infty} |\hat{f}(\xi) \cdot e^{-4\pi^2\xi^2 t} - \hat{f}(\xi)|^2 d\xi$$

Observe that

$$\left| \hat{f}(\xi) e^{-4\pi^2\xi^2 t} - \hat{f}(\xi) \right|^2$$

$$= |\hat{f}(\xi)|^2 \cdot \left| e^{-4\pi^2\xi^2 t} - 1 \right|^2$$

$$\leq |\hat{f}(\xi)|^2 .$$

Hence by Dominated convergence Thm,

$$\lim_{t\to 0} \int_{-\infty}^{\infty} \left| \hat{f}(\xi) e^{-4\pi^2\xi^2 t} - \hat{f}(\xi) \right|^2 d\xi$$

$$= \int_{-\infty}^{\infty} \lim_{t\to 0} \left| \hat{f}(\xi) e^{-4\pi^2\xi^2 t} - \hat{f}(\xi) \right|^2 d\xi$$

$$= 0 .$$

That is,

$$\lim_{t\to 0} \int_{\mathbb{R}} |u(x,t) - f(x)|^2 dx = 0 .$$

$\boxed{/\!/}$

Q: Is $u = f * \mathcal{H}_t$ the unique solution to the heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} & \text{on } \mathbb{R} \times \mathbb{R}_+, \\ u(x,0) = f(x). \end{cases}$$

The answer is no!

Example 1. Let $u(x,t) = \dfrac{x}{t} \mathcal{H}_t(x)$ on $\mathbb{R} \times \mathbb{R}_+$.

Check: • $u \in C^\infty(\mathbb{R} \times \mathbb{R}_+)$

$$(*) \begin{cases} \bullet \ \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} & \text{on } \mathbb{R} \times \mathbb{R}_+ \\ \bullet \ \lim\limits_{t \to 0} u(x,t) = 0. \end{cases}$$

However $u \equiv 0$ is also a solution of $(*)$.

So $(*)$ has more than one solution.

Def. Given a function $u(x,t)$ on $\mathbb{R}\times\mathbb{R}_+$, we say that $u(\cdot,t)$ belongs to $S(\mathbb{R})$ uniformly in t if $u(\cdot,t)\in S(\mathbb{R})$ for all $t>0$, and moreover, for each $T>0$, and each $k,\ell\geq 0$.

$$\sup_{\substack{x\in\mathbb{R}\\ 0<t<T}} \left| x^k \cdot \frac{\partial^\ell u(x,t)}{\partial x^\ell} \right| < \infty .$$

Check: $\frac{x}{t}\cdot \mathcal{H}_t(x)$ does not belongs to $S(\mathbb{R})$ uniformly in t.

Because $$\sup_{\substack{x\in\mathbb{R}\\ 0<t<T}} \left|\frac{x}{t}\,\mathcal{H}_t(x)\right| \geq \sup_{0<t<T} \left|\frac{\sqrt{t}}{t}\,\mathcal{H}_t(\sqrt{t})\right|$$

$$= \sup_{0<t<T} \frac{1}{\sqrt{4\pi}\, t}\cdot e^{-\frac{1}{4}}$$

$$= +\infty .$$

Thm (Uniqueness).

Let $u \in C^\infty(\mathbb{R} \times \mathbb{R}_+) \cap C(\mathbb{R} \times [0,\infty))$.
Suppose u satisfies the following properties.

① $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$ on $\mathbb{R} \times \mathbb{R}_+$;

② $u(x,0) = 0$ for all $x \in \mathbb{R}$;

③ $u(\cdot, t) \in S(\mathbb{R})$ uniformly in t.

Then $u(x,t) = 0$ on $\mathbb{R} \times \mathbb{R}_+$.

Pf. (Energy method)

Define for $t \geq 0$,

$$E(t) = \int_{-\infty}^{\infty} |u(x,t)|^2 \, dx$$

In particular, $E(0) = 0$, $E(t) \geq 0$ for $t \geq 0$.

Observe that

$$\frac{d}{dt}E(t) = \frac{d}{dt}\int_{-\infty}^{\infty} |u(x,t)|^2\,dx$$

$$\overset{\text{By (DCT)}}{=\!=} \int_{-\infty}^{\infty} \frac{d}{dt}|u(x,t)|^2\,dx$$

Here we need to use $u \in C^\infty(\mathbb{R}\times\mathbb{R}_+)$ and $u(\cdot,t) \in \mathcal{S}(\mathbb{R})$ unif. in t.

$$= \int_{-\infty}^{\infty} \frac{d}{dt}\left(u(x,t)\,\overline{u(x,t)}\right)dx$$

$$= \int_{-\infty}^{\infty} \left(\partial_t u(x,t)\right)\cdot\overline{u(x,t)} + u(x,t)\,\overline{\partial_t u(x,t)}\,dx$$

$$= \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2}\cdot\overline{u} + u\cdot\frac{\partial^2\overline{u}}{\partial x^2}\,dx$$

$$\overset{\text{Integration by Parts}}{=} \frac{\partial u}{\partial x}\overline{u}\Big|_{x=-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{\partial u}{\partial x}\cdot\frac{\partial\overline{u}}{\partial x}\,dx$$

$$+\ u\cdot\frac{\partial\overline{u}}{\partial x}\Big|_{x=-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{\partial u}{\partial x}\cdot\frac{\partial\overline{u}}{\partial x}\,dx$$

$$= -2\int_{-\infty}^{\infty} \frac{\partial u}{\partial x} \cdot \frac{\partial \bar{u}}{\partial x}\, dx$$

$$= -2\int_{-\infty}^{\infty} \left| \frac{\partial u}{\partial x} \right|^2 dx$$

$$\leq 0.$$

Hence $E(t)$ is a non-increasing function on $(0, \infty)$.

But $E(0) = 0$, so $E(t) \leq 0$ for $t > 0$.

However, by definition $E(t) \geq 0$.

As a consequence,

$$E(t) \equiv 0 \text{ for all } t \geq 0.$$

That is

$$\int_{-\infty}^{\infty} |u(x,t)|^2 dx \equiv 0.$$

$$\Rightarrow \quad u(x,t) \equiv 0. \qquad \square$$

Prop. Let $f \in S(\mathbb{R})$ and let

$$u(x,t) = f * \mathcal{H}_t(x).$$

Then $u(\cdot, t) \in S(\mathbb{R})$ uniformly in t in the sense that

$$\sup_{\substack{x \in \mathbb{R} \\ 0<t<T}} \left| x^k \frac{\partial^\ell u(x,t)}{\partial x^\ell} \right| < \infty \qquad (**)$$

for any $T>0$ and $k, \ell \geq 0$.

Pf. Without loss of generality, we only prove $(**)$ in the case when $k=1, \ell=1$.

Let $T>0$.

$$\frac{\partial u(x,t)}{\partial x} \xrightarrow{\mathcal{F}} (2\pi i \xi)\, \hat{u}(\xi, t)$$

$$= 2\pi i \xi \cdot \hat{f}(\xi) \cdot e^{-4\pi^2 \xi^2 t}.$$

$$(-2\pi i x)\cdot \frac{\partial u(x,t)}{\partial x} \xrightarrow{\mathcal{F}} \frac{d\left(2\pi i \xi \hat{f}(\xi) e^{-4\pi^2\xi^2 t}\right)}{d\xi}$$

By Inversion formula,

$$-2\pi i x \cdot \frac{\partial u(x,t)}{\partial x} = \int_{\mathbb{R}} \frac{d\left(2\pi i \xi \hat{f}(\xi) e^{-4\pi^2\xi^2 t}\right)}{d\xi} e^{2\pi i \xi x} \, d\xi$$

So

$$\sup_{\substack{x\in\mathbb{R}\\ 0<t<T}} 2\pi \left| x \cdot \frac{\partial u}{\partial x}(x,t) \right| \le \sup_{\substack{x\in\mathbb{R}\\ 0<t<T}} \int_{\mathbb{R}} \underbrace{\left| \frac{d\left(2\pi i \xi \hat{f}(\xi) e^{-4\pi^2\xi^2 t}\right)}{d\xi} \right|}_{(***)} d\xi$$

$$= \sup_{0<t<T} \int_{\mathbb{R}} |(***)| \, d\xi$$

Notice that

$$(***) = 2\pi i \Big[\big(\xi \hat{f}(\xi)\big)' e^{-4\pi^2\xi^2 t} + \xi \hat{f}(\xi) \cdot e^{-4\pi^2\xi^2 t} \cdot (-8\pi\xi t) \Big]$$

Hence

$$|(***)| \leq 2\pi \cdot \big| \big(\xi \hat{f}(\xi)\big)' \big| + 16\pi^2 |\xi^2 \hat{f}(\xi)| \cdot t.$$

Hence

$$\sup_{0<t<T} \int_{\mathbb{R}} |(***)| \, d\xi \leq 2\pi \int \big|\big(\xi \hat{f}(\xi)\big)'\big| \, d\xi + 16\pi^2 \int_{\mathbb{R}} |\xi^2 \hat{f}(\xi)| \, d\xi \cdot T$$

$< \infty.$